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Asymptotic approximation by de la Vallée Poussin means and their derivatives

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Abstract

The concern of this paper is the study of local approximation properties of the de la Vallée Poussin means V_n. We derive the complete asymptotic expansion of the operators V_n and their derivatives as n tends to infinity. It turns out that the appropriate representation is a series of reciprocal factorials. All coefficients are calculated explicitly.

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1. Introduction

The de la Vallée Poussin means V_n $(n = 1, 2, \ldots)$ of a 2π-periodic integrable function f are defined by

$$V_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\omega_n(x-t)\,dt \quad (x \in [0, 2\pi)), \tag{1}$$

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with the kernel ω_n given by

$$\omega_n(x) = \frac{(n!)^2}{(2n)!}\left(2\cos\frac{x}{2}\right)^{2n} = \sum_{v=-n}^{n} \frac{(n!)^2}{(n-v)!\,(n+v)!} e^{ivx} \quad (x \in \mathbb{R}) \tag{2}$$

(see, e.g., [12, Section 2.5.2] or [12, pp. 299–300]). The operators V_n are trigonometric analogues of the Bernstein polynomials. They are shape-preserving trigonometric convolution operators [21].

The Voronovskaja-type formula

$$\lim_{n\to\infty} n(V_n(f;x) - f(x)) = f^{(2)}(x) \tag{3}$$

for all integrable 2π-periodic functions f admitting a derivative of second order at x is due to Natanson (see [20, Chapter 10, Section 3, Satz 3] and the original paper cited there; cf. also [17, Problem 29, p. 134]).

Formula (3) is a trigonometric analogue of Voronovskaja's classical theorem [23] for the Bernstein polynomials which was generalized by Bernstein [9] for higher order differentiable functions.

In the same manner as Bernstein, among other things, Lee and Osman [18] extended Natanson's result (3).

Let $C_{2\pi}$ denote the class of continuous 2π-periodic functions. For an even 2π-periodic integrable function φ, its trigonometric moment of order $2j$ ($j = 0, 1, 2, \ldots$), is defined by

$$M_{2j}(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(2\sin\frac{t}{2}\right)^{2j}\varphi(t)\,dt. \tag{4}$$

Furthermore, for $s, v \in \mathbb{N}_0$, let

$$a_{s,v}(\varphi) = \sum_{j=v}^{s} \frac{(-1)^{j+v}}{(2j)!} t(2j, 2v)\, M_{2j}(\varphi), \tag{5}$$

where $t(\cdot,\cdot)$ denote the central factorial numbers of the first kind (see, e.g., [22, p. 213]). Further properties and applications can be found in [13,14]. Recall that the $t(\cdot,\cdot)$ are the coefficients in the expansion

$$x^{[n]} = \sum_{j=0}^{n} t(n,j) x^j \quad (n = 0, 1, 2, \ldots),$$

where the central factorial polynomials $x^{[n]}$ are defined as

$$x^{[0]} = 1 \quad \text{and} \quad x^{[n]} = x(x + 1 - n/2)^{\overline{n-1}} \quad (n \in \mathbb{N}).$$

Throughout the paper $n^{\bar{k}}$ resp. $n^{\underline{k}}$ denotes the rising factorial $n^{\bar{k}} = n(n+1)\cdots(n+k-1)$, $n^{\bar{0}} = 1$ resp. falling factorial $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$, $n^{\underline{0}} = 1$.

The above-mentioned result of Lee and Osman [18, Corollary 2.1] states that, if $f \in C_{2\pi}$ and its derivatives up to order $2s$ exist at $x \in (-\pi, \pi)$, there holds

$$\lim_{n\to\infty} n^s\left(V_n(f;x) - \sum_{v=0}^{s-1} a_{s,v}(\omega_n) f^{(2v)}(x)\right) = \frac{f^{(2s)}(x)}{s!}. \tag{6}$$

However, the asymptotic relation (6) does not give much insight into the asymptotic behaviour of operators (1). This motivates a further treatment. The purpose of this paper is to continue the work of Lee and Osman in order to derive the complete asymptotic expansion for the sequence of de la Vallée Poussin means V_n in the form

$$V_n(f;x) \sim f(x) + \sum_{k=1}^{\infty} \frac{c_k(f;x)}{(n+1)^{\bar{k}}} \quad (n \to \infty), \tag{7}$$

provided $f \in C_{2\pi}$ possesses derivatives of sufficiently high order at x. It turns out that the appropriate representation of expansion (7) is a series of reciprocal factorials. Formula (7) means that, for all $q \in \mathbb{N}$,

$$V_n(f;x) = f(x) + \sum_{k=1}^{q} \frac{c_k(f;x)}{(n+1)^{\bar{k}}} + o(n^{-q})$$

as $n \to \infty$. Result (3) of Natanson is the special case $q = 1$ with $c_1(f;x) = f''(x)$. We give explicit expressions for all coefficients $c_k(f;x) (k = 1, 2, \ldots)$.

We remark that in [1–4,6,7] the author gave analogous results for the Meyer-König and Zeller operators, for the operators of Bleimann, Butzer and Hahn, the Bernstein–Kantorovich operators, the Bernstein–Durrmeyer operators, and the operators of Balázs and Szabados, respectively. Asymptotic expansions of multivariate operators can be found in [5,8].

After completion of the manuscript the author learned by personal communication from Prof. Butzer that formula (8) (i.e., the special case $r = 0$ of Theorem 2) was previously found by Bleimann and Stark [10] under the stronger assumption that $f \in C_{2\pi}^{2s}$. Our results require only local smoothness of f.

2. The main results

As first main result we obtain the complete asymptotic expansion of the de la Vallée Poussin means V_n.

Theorem 1. *Let $s \in \mathbb{N}$ and $x \in (-\pi, \pi)$. If $f \in C_{2\pi}$ and its derivatives up to order $2s$ exist at x, the de la Vallée Poussin means V_n satisfy the asymptotic relation*

$$V_n(f;x) = f(x) + \sum_{k=1}^{s} \frac{c_k(f;x)}{(n+1)^{\bar{k}}} + o(n^{-s}) \tag{8}$$

as $n \to \infty$, where the coefficients $c_k(f;x)$ are given by

$$c_k(f;x) = \frac{(-1)^k}{k!} \sum_{v=0}^{k} (-1)^v t(2k, 2v) f^{(2v)}(x) \tag{9}$$

and $t(\cdot,\cdot)$ denote the central factorial numbers of the first kind.

For the convenience of the reader we calculate the explicit form of the initial coefficients $c_k(f;x)$ in expansion (8):

$$
\begin{aligned}
c_0(f;x) &= f(x),\\
c_1(f;x) &= f^{(2)}(x),\\
c_2(f;x) &= \tfrac{1}{2}(f^{(2)}(x)+f^{(4)}(x)),\\
c_3(f;x) &= \tfrac{1}{6}(4f^{(2)}(x)+5f^{(4)}(x)+f^{(6)}(x)),\\
c_4(f;x) &= \tfrac{1}{24}(36f^{(2)}(x)+49f^{(4)}(x)+14f^{(6)}(x)+f^{(8)}(x)),\\
c_5(f;x) &= \tfrac{1}{120}(576f^{(2)}(x)+820f^{(4)}(x)+273f^{(6)}(x)+30f^{(8)}(x)+f^{(10)}(x)).
\end{aligned}
$$

Concerning simultaneous approximation it is well known that $\lim_{n\to\infty}(d/dx)V_n(f;x)=f'(x)$ if $f\in C_{2\pi}$ possesses a derivative of first order at x ([20, Chapter 10, Section 3, Satz 4]; cf. [11]). In this direction, we derive the complete asymptotic expansion of the differentiated de la Vallée Poussin means $V_n^{(r)}$ $(r=0,1,2,\ldots)$.

Theorem 2. *Let $r\in\mathbb{N}_0, s\in\mathbb{N}$ and $x\in(-\pi,\pi)$. If $f\in C_{2\pi}$ and its derivatives up to order $2(r+s)$ exist at x, the differentiated de la Vallée Poussin means $V_n^{(r)}$ satisfy the asymptotic relation*

$$V_n^{(r)}(f;x)=f^{(r)}(x)+\sum_{k=1}^{s}\frac{c_k(f^{(r)};x)}{(n+1)^{\bar{k}}}+o(n^{-s}) \tag{10}$$

as $n\to\infty$, where the coefficients $c_k(f;x)$ are given by Eq. (9).

Remark 1. Note that, in Theorem 2, we propose only local smoothness conditions on f. If we assume, in addition, that $f^{(r)}$ exists on $(-\pi,\pi)$, Theorem 2 would be an immediate corollary of Theorem 1, since it is well known that

$$V_n^{(r)}(f;x)=V_n(f^{(r)};x)$$

if $f^{(r)}\in C_{2\pi}$ (see, e.g., [12, Proposition 1.1.15]).

3. Auxiliary results

First, we determine the trigonometric moments of the functions ω_n as defined by Eq. (2).

Lemma 3. *For $j,n=0,1,2,\ldots$, we have*

$$M_{2j}(\omega_n)=\binom{2j}{j}\binom{j+n}{j}^{-1}. \tag{11}$$

Proof. By definition (4), we obtain

$$M_{2j}(\omega_n) = \frac{2^{2j+2n}}{2\pi}\binom{2n}{n}^{-1} 2\int_{-\pi/2}^{\pi/2} \sin^{2j} t \cos^{2n} t \, dt$$
$$= \frac{2^{2j+2n}}{\pi}\binom{2n}{n}^{-1} B\left(j+\frac{1}{2}, n+\frac{1}{2}\right)$$

with the beta function B, and Eq. (11) follows by a simple calculation. □

In the following, for $f \in C_{2\pi}$, let f_x be defined as

$$f_x(t) = f(t)\sin(x-t). \qquad (12)$$

Formula (3) is a trigonometric analogue of Voronovskaja's classical theorem [23] (cf. [15, Chapter 10, Section 3, Theorem 3.1]) for the Bernstein polynomials which was generalized by Bernstein [9] (cf. [19, Section 1.6.1, Eq. (4)]) for higher order differentiable functions.

Lemma 4. *For $f \in C_{2\pi}$, we have*

$$V_n'(f;x) = \frac{-n^2}{2n-1} V_{n-1}(f_x;x) \qquad (13)$$

and

$$V_n''(f;x) = -n^2(V_n(f;x) - V_{n-1}(f;x)). \qquad (14)$$

Proof. By Butzer and Nessel [12, Proposition 1.1.14], there holds

$$V_n^{(r)}(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\omega_n^{(r)}(x-t)\, dt \quad (r = 0, 1, 2, \ldots)$$

and Lemma 4 follows by straightforward computation. Formula (14) also can be found in [12, Eq. (2.5.21)]. □

Lemma 5. *If $f \in C_{2\pi}$ and its derivatives up to order $2k (k \in \mathbb{N})$ exist at x, the coefficient $c_k(f_x;x)$, as defined in Eq. (9), for f_x is given by*

$$c_k(f_x;x) = \frac{(-1)^k}{k!}\sum_{\mu=0}^{k-1}(-1)^{\mu} f^{(2\mu+1)}(x)\sum_{v=\mu+1}^{k}\binom{2v}{2\mu+1} t(2k, 2v).$$

Proof. Lemma 5 follows by Leibniz rule. □

Lemma 6. *Suppose $k > -1/2$. Then, we have*

$$\frac{n}{2n-1} = \frac{1}{2} + \frac{1}{4}\sum_{j=0}^{\infty}\frac{(k+\frac{1}{2})^{\overline{j}}}{(n+k)^{\overline{j+1}}} \quad (n > 1/2).$$

Proof. Let $k > -1/2$ and $n > 1/2$. Then, we have

$$\frac{2n}{2n-1} = \frac{1 - \frac{k}{n+k}}{1 - \frac{k+\frac{1}{2}}{n+k}} = 1 + \frac{1}{2}\sum_{\ell=0}^{\infty} \frac{(k+\frac{1}{2})^{\ell}}{(n+k)^{\ell+1}}.$$

Using the well-known relation

$$\frac{1}{z^{\ell+1}} = \sum_{j=\ell}^{\infty} |S_j^{\ell}| \frac{1}{z^{\overline{j+1}}} \quad (\Re z > 0)$$

(see, e.g., [16, Example 1, p. 214]), where S_j^{ℓ} denote the Stirling numbers of the first kind, we obtain

$$\frac{2n}{2n-1} = 1 + \frac{1}{2}\sum_{j=\ell}^{\infty} \frac{1}{(n+k)^{\overline{j+1}}} \sum_{\ell=0}^{j} |S_j^{\ell}| \left(k + \frac{1}{2}\right)^{\ell}$$

and Lemma 6 follows since $\sum_{\ell=0}^{j} |S_j^{\ell}|(k+\frac{1}{2})^{\ell} = (k+\frac{1}{2})^{\overline{j}}$. □

Lemma 7. *The central factorial numbers of the first kind satisfy, for* $k \in \mathbb{N}$ *and* $\mu = 0, 1, 2, \ldots,$ *the both identities*

$$\sum_{v=0}^{k} \binom{2v}{2\mu+1} t(2k, 2v) = k((k-1)t(2k-2, 2\mu+2) + 2t(2k-2, 2\mu)), \tag{15}$$

$$\sum_{j=1}^{k} \frac{(-1)^j}{j!}(j+\frac{1}{2})^{\overline{k-j}} \sum_{v=0}^{j} \binom{2v}{2\mu+1} t(2j, 2v) = \frac{2(-1)^k}{(k-1)!} t(2k, 2\mu+2). \tag{16}$$

Proof. We have

$$\sum_{v=0}^{k} \binom{2v}{2\mu+1} t(2k, 2v) = \frac{1}{(2\mu+1)!}\left(\frac{d}{dx}\right)^{2\mu+1} x^{[2k]}\Bigg|_{x=1}.$$

The latter expression is the coefficient of $x^{2\mu+1}$ in the polynomial

$$(x+1)^{[2k]} = \left(1 + \frac{k}{x}\right)x^{[2k]} + k\left(x + (2k-1) + \frac{k^2-k}{x}\right)x^{[2k-2]}$$

which comes out to be

$$kt(2k, 2\mu+2) + kt(2k-2, 2\mu) + (k^3 - k^2)t(2k-2, 2\mu+2).$$

Hence,

$$\sum_{v=0}^{k} \binom{2v}{2\mu+1} t(2k, 2v)$$
$$= k(t(2k, 2\mu+2) + t(2k-2, 2\mu) + ((k-1)^2 + k - 1)t(2k-2, 2\mu+2)).$$

The recurrence formula for the central factorial numbers of the first kind (see [22, Eq. (25), p. 214]) yields

$$t(2k, 2\mu + 2) + (k-1)^2 t(2k-2, 2\mu+2) = t(2k-2, 2\mu),$$

which implies Eq. (15).

By Eq. (15), formula (16) is equivalent to

$$\sum_{j=1}^{k} \frac{(-1)^j}{(j-1)!}\left(k - \frac{1}{2}\right)^{\underline{k-j}} ((j-1)t(2j-2, 2\mu+2) + 2t(2j-2, 2\mu))$$
$$= \frac{2(-1)^k}{(k-1)!} t(2k, 2\mu+2), \tag{17}$$

which we will prove by mathematical induction. Eq. (17) obviously holds for $k = 1$. Assume that it is true for an arbitrary $k \in \mathbb{N}$. Then, we have

$$\sum_{j=1}^{k+1} \frac{(-1)^j}{(j-1)!}\left(k + 1 - \frac{1}{2}\right)^{\underline{k+1-j}} ((j-1)t(2j-2, 2\mu+2) + 2t(2j-2, 2\mu))$$
$$= \left(k + \frac{1}{2}\right) \sum_{j=1}^{k} \frac{(-1)^j}{(j-1)!}\left(k - \frac{1}{2}\right)^{\underline{k-j}} ((j-1)t(2j-2, 2\mu+2)$$
$$+ 2t(2j-2, 2\mu)) + \frac{(-1)^{k+1}}{k!}(kt(2k, 2\mu+2) + 2t(2k, 2\mu))$$
$$= (2k+1)\frac{(-1)^k}{(k-1)!} t(2k, 2\mu+2)$$
$$+ \frac{(-1)^{k+1}}{k!}(kt(2k, 2\mu+2) + 2t(2k, 2\mu))$$
$$= \frac{2(-1)^k}{k!}(k^2 t(2k, 2\mu+2) - t(2k, 2\mu))$$
$$= -\frac{2(-1)^k}{k!} t(2k+2, 2\mu+2),$$

where we again used the recurrence formula [22, Eq. (25), p. 214]. This completes the proof of Lemma 7. □

Now we show, that Theorem 2 holds in the special case $r = 1$.

Proposition 8. *Let $s \in \mathbb{N}$ and $x \in (-\pi, \pi)$. If $f \in C_{2\pi}$ and its derivatives up to order $2s+2$ exist at x, the differentiated de la Vallée Poussin means V_n' satisfy the asymptotic relation*

$$V_n'(f; x) = f'(x) + \sum_{k=1}^{s} \frac{c_k(f'; x)}{(n+1)^{\bar{k}}} + o(n^{-s})$$

as $n \to \infty$, where the coefficients $c_k(f; x)$ are given by Eq. (9).

Proof. Combining Eq. (13) with Theorem 1 yields

$$V_n'(f;x) = \frac{-n^2}{2n-1} V_{n-1}(f_x;x) = \frac{-n}{2n-1} \sum_{k=1}^{s+1} \frac{c_k(f_x;x)}{(n+1)^{\overline{k-1}}} + o(n^{-s})$$

as $n \to \infty$ and application of Lemma 6 yields

$$\begin{aligned} V_n'(f;x) &= -\frac{1}{2}\sum_{k=1}^{s+1} \frac{c_k(f_x;x)}{(n+1)^{\overline{k-1}}} - \frac{1}{4}\sum_{k=1}^{s}\sum_{j=0}^{s-k} \left(k+\frac{1}{2}\right)^{\overline{j}} \frac{c_k(f_x;x)}{(n+1)^{\overline{k+j}}} + o(n^{-s}) \\ &= -\frac{1}{2}\sum_{k=0}^{s} \frac{c_{k+1}(f_x;x)}{(n+1)^{\bar{k}}} - \frac{1}{4}\sum_{k=1}^{s} \frac{1}{(n+1)^{\bar{k}}} \sum_{j=1}^{k} c_j(f_x;x)\left(j+\frac{1}{2}\right)^{\overline{k-j}} + o(n^{-s}) \\ &= \sum_{k=0}^{s} \frac{d_k(f;x)}{(n+1)^{\bar{k}}} + o(n^{-s}) \quad (n \to \infty), \end{aligned}$$

say, where

$$d_k(f;x) = -\frac{1}{2} c_{k+1}(f_x;x) - \frac{1}{4}\sum_{j=1}^{k} \left(j+\frac{1}{2}\right)^{\overline{k-j}} c_j(f_x;x)$$

with the convention that a sum is to be read as 0 if the lower index is greater than the upper index. We have to show that

$$d_k(f;x) = c_k(f';x) \quad (k = 0, \dots, s). \tag{18}$$

For $k = 0$, we have

$$d_0(f;x) = -\tfrac{1}{2} c_1(f_x;x) = -\tfrac{1}{2} f_x''(x) = f'(x) = c_0(f';x).$$

Now, let $k \geqslant 1$. By Lemma 5, we have

$$\begin{aligned} d_k(f;x) = &\frac{(-1)^k}{2(k+1)!} \sum_{\mu=0}^{k} (-1)^\mu f^{(2\mu+1)}(x) \sum_{\nu=\mu+1}^{k+1} \binom{2\nu}{2\mu+1} t(2k+2, 2\nu) \\ &- \frac{1}{4}\sum_{j=1}^{k} \left(j+\frac{1}{2}\right)^{\overline{k-j}} \frac{(-1)^j}{j!} \sum_{\mu=0}^{j-1} (-1)^\mu f^{(2\mu+1)}(x) \\ &\times \sum_{\nu=\mu+1}^{j} \binom{2\nu}{2\mu+1} t(2j, 2\nu) \end{aligned}$$

and, by Lemma 7, we obtain

$$d_k(f;x) = \frac{(-1)^k}{2k!}\sum_{\mu=0}^{k} (-1)^\mu f^{(2\mu+1)}(x)(kt(2k,2\mu+2) + 2t(2k,2\mu))$$

$$- \frac{(-1)^k}{2(k-1)!}\sum_{\mu=0}^{k-1} (-1)^\mu f^{(2\mu+1)}(x)\, t(2k,2\mu+2)$$

$$= \frac{(-1)^k}{k!}\sum_{\mu=0}^{k} (-1)^\mu f^{(2\mu+1)}(x)\, t(2k,2\mu) = c_k(f';x).$$

In view of Eq. (18) this completes the proof of the proposition. □

4. Proof of the main theorems

Proof of Theorem 1. By the result of Lee and Osman (6), there holds

$$V_n(f;x) = \sum_{v=0}^{s-1} a_{s,v}(\omega_n) f^{(2v)}(x) + \frac{f^{(2s)}(x)}{s!n^s} + o(n^{-s}) \tag{19}$$

as $n \to \infty$, provided $f \in C_{2\pi}$ and its derivatives up to order $2s$ exist at x. By Eq. (5) and Lemma 3, we have

$$a_{s,v}(\omega_n) = \sum_{k=v}^{s} \frac{(-1)^{k+v}}{k!\,(n+1)^{\bar{k}}}\, t(2k,2v),$$

and inserting this into Eq. (19) we obtain

$$V_n(f;x) = \sum_{k=0}^{s} \frac{(-1)^k}{k!\,(n+1)^{\bar{k}}} \sum_{v=0}^{k} (-1)^v t(2k,2v) f^{(2v)}(x) + o(n^{-s})$$

as $n \to \infty$. This completes the proof of Theorem 1. □

Proof of Theorem 2. By Theorem 1 and the proposition, the assertion of Theorem 2 is true for $r = 0$ and $r = 1$. We proceed by mathematical induction. Assume that Theorem 2 is true for an integer $r \geqslant 0$, i.e.

$$V_n^{(r)}(f;x) = f^{(r)}(x) + \sum_{k=1}^{s+2} \frac{c_k(f^{(r)};x)}{(n+1)^{\bar{k}}} + o(n^{-(s+2)}) \tag{20}$$

as $n \to \infty$, provided $f \in C_{2\pi}$ and its derivatives up to order $2(r+s+2)$ exist at x. Then, by Lemma 4, we have

$$V_n^{(r+2)}(f;x) = -n^2(V_n^{(r)}(f;x) - V_{n-1}^{(r)}(f;x)).$$

Taking advantage of Eq. (20) we obtain

$$\begin{aligned} V_n^{(r+2)}(f;x) &= -n^2 \sum_{k=1}^{s+2} c_k(f^{(r)};x)\left(\frac{1}{(n+1)^{\bar{k}}} - \frac{1}{n^{\bar{k}}}\right) + o(n^{-s}) \\ &= n \sum_{k=1}^{s+1} \frac{k c_k(f^{(r)};x)}{(n+1)^{\bar{k}}} + o(n^{-s}) \\ &= \sum_{k=0}^{s} \frac{(k+1)c_{k+1}(f^{(r)};x) - k^2 c_k(f^{(r)};x)}{(n+1)^{\bar{k}}} + o(n^{-s}) \end{aligned}$$

as $n \to \infty$. By definition (9), we have

$$\begin{aligned} &(k+1)c_{k+1}(f^{(r)};x) - k^2 c_k(f^{(r)};x) \\ &\quad = \frac{(-1)^{k+1}}{k!} \sum_{v=0}^{k+1} (-1)^v (t(2k+2,2v) + k^2 t(2k,2v)) f^{(r+2v)}(x) \\ &\quad = \frac{(-1)^{k+1}}{k!} \sum_{v=1}^{k+1} (-1)^v t(2k,2v-2) f^{(r+2v)}(x) = c_k(f^{(r+2)};x), \end{aligned}$$

where we used the recurrence formula [22, Eq. (25), p. 214]. This completes the proof of Theorem 2. □

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